Spontaneous chiral symmetry breaking in QCD: a finite-size scaling study on the lattice

Silvia Necco

IFIC/Valencia University

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Introduction

- On finite volume $\rightarrow$ spontaneous symmetry breaking does not occur
  But: the formation of a condensate Σ in infinite volume leaves signs in a finite box
  $\rightarrow$ finite-size scaling effects are predicted by chiral perturbation theory

  Leutwyler and Smilga (1992)

  For $m\Sigma V \ll 1$:

  $$\frac{\Sigma(m, V)}{\Sigma} \propto m\Sigma V$$

  Predictions in terms of the same low-energy constants appearing in the infinite volume

- $\rightarrow$ Σ can be extracted from finite-size scaling study
  $\rightarrow$ Ginsparg-Wilson fermions

  P. Hernández et al (1999), T. DeGrand(2001), P.Hasenfratz et al (2002)

Quark condensate with Ginsparg-Wilson fermions

Euclidean lattice, $V = L^4$, lattice spacing a, $\psi = (\psi_1, ..., \psi_{N_f})$
Neuberger Dirac operator

Ginsparg & Wilson (1982), Neuberger (1998)

- Topological charge density:

  $a^4 q(x) = -\frac{a}{2}\mathrm{tr}\left[\gamma_5 D(x,x)\right] \rightarrow Q = a^4 \sum_x q(x)$
- Index theorem: in a given gauge field background:

$$Q = n_+ - n_-$$

n_+ (n_-) number of zero-modes of D with positive (negative) chirality

- Quark condensate in the chiral limit:

$$\langle\overline{\psi}\psi\rangle = \lim_{a\to 0} Z_S \langle\overline{\psi}\tilde{\psi}\rangle$$

$$\tilde{\psi} = \left(1 - \tfrac{\bar{a}}{2} D\right)\psi$$

- At finite quark mass: UV divergences

$$-\frac{Z_S\langle\overline{\psi}\tilde{\psi}\rangle}{N_f} = b_1 m + b_2 m^3 + \{\text{finite terms}\}\,.$$

$b_1 \propto 1/a^2,\ b_2 \propto \ln(a)$

- The condensate can be defined at fixed topological charge $\nu = |Q|$: same UV divergences

$$-\frac{\langle\overline{\psi}\tilde{\psi}\rangle_\nu}{N_f} = \frac{\nu}{Vm} + \chi_\nu$$

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$1/m$ divergence: zero-modes contributions

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$$-\frac{\langle\overline{\psi}\tilde{\psi}\rangle_\nu}{N_f} = \frac{\nu}{Vm} + \underbrace{\chi_\nu}$$

$$\hat{\chi}_\nu = Z_S\chi_\nu = b_1 m + b_2 m^3 + \{\text{finite terms}\}$$

Z_S, b_1, b_2 topology independent

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b_1, b_2 are volume independent and suppressed by $1/V$ with respect to the finite terms

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$$-\frac{\langle \overline{\psi}\tilde{\psi} \rangle_\nu}{N_f} = \frac{\nu}{Vm} + \chi_\nu$$

$$\hat{\chi}_\nu = Z_S \chi_\nu = b_1 m + b_2 m^3 + \{\text{finite terms}\}$$

$\rightarrow (\hat{\chi}_{\nu_1} - \hat{\chi}_{\nu_2})$ is unambiguously defined at finite quark mass

An alternative point of view:
spectral density of the massive Dirac operator:
$|\lambda_k|^2$: eigenvalues of $D^\dagger D$ in the sector without zero modes

$$\rho(\lambda) = \frac{1}{V} \sum_k \langle \delta(\lambda - |\lambda_k|) \rangle$$

Banks and Casher (1980)

- has a well-defined thermodynamic limit
- has a well-defined continuum limit

Del Debbio et al (2005)

- can be defined at fixed topology $\rightarrow \quad \rho_\nu(\lambda)$

$\rightarrow$ integrals of $\rho(\lambda)$ with appropriately chosen probe functions can be computed on the lattice. Example ($\hat{\lambda} = \lambda/Z_s$, $\hat{m} = m/Z_S$):

$$\hat{\chi}_\nu = 2\hat{m} \int_0^\infty \frac{1}{\hat{\lambda}^2 + \hat{m}^2} \hat{\rho}_\nu(\hat{\lambda}) d\hat{\lambda}$$

The UV divergences in the condensate are due to the integration up to ∞

$$\hat{\tau}_\nu(\hat{\lambda}_{\min}, \hat{\lambda}_{\max}) = 2\hat{m} \int_{\hat{\lambda}_{\min}}^{\hat{\lambda}_{\max}} \frac{1}{\hat{\lambda}^2 + \hat{m}^2} \hat{\rho}_\nu(\hat{\lambda}) d\hat{\lambda}$$

- $\hat{\tau}_\nu(\hat{\lambda}_{\min}, \hat{\lambda}_{\max})$ has a well defined continuum limit, if $(\hat{\lambda}_{\min}, \hat{\lambda}_{\max})$ are kept fixed when $a \to 0$
- $\to$ in particular our observable will be

$$\underbrace{\hat{\tau}_{\nu_1}(\hat{\lambda}_{\min 1}, \infty) - \hat{\tau}_{\nu_2}(\hat{\lambda}_{\min 2}, \infty)} = \hat{\chi}_{\nu_1} - \hat{\chi}_{\nu_2} - \left[\hat{\tau}_{\nu_1}(0, \hat{\lambda}_{\min 1}) - \hat{\tau}_{\nu_2}(0, \hat{\lambda}_{\min 2})\right]$$

Strategy: this quantity

L. Giusti, S.N. (hep-lat/0701023)

- is UV-finite
- can be computed using stable numerical estimators $\to$ low-mode averaging

L. Giusti, C. Hölbling, M. Lüscher, H. Wittig (2003)

L. Giusti, P. Hernández, M. Laine, P. Weisz, H. Wittig (2004)

- can be matched directly with chiral effective theory at NLO $\to$ extraction of the low-energy constant

Quark condensate in the chiral effective theory

- Predictions in the chiral effective theory at finite volume ($L \gg 1/\Lambda_{\rm QCD}$) are possible

  Gasser and Leutwyler (1987)

- Predictions in sectors of fixed topology are possible

  Leutwyler and Smilga (1992)

- LO prediction for the quenched spectral density ($\zeta = \hat{\lambda}\Sigma V$)

  Osborn et al (1999), Damgaard et al (2002)

$$\overline{\rho}_\nu(\zeta) = \frac{1}{\Sigma}\tilde{\rho}_\nu(\zeta/\Sigma V) = \frac{\zeta}{2}\left[J_\nu(\zeta)^2 - J_{\nu+1}(\zeta)J_{\nu-1}(\zeta)\right]$$

  $\rightarrow$ match with $\hat{\rho}(\hat{\lambda})$ for $\hat{\lambda} \ll \Lambda_{\rm QCD} \ll \frac{1}{a}$

- NLO corrections are known by means of the ϵ- expansion: same analytical form as LO, with $\Sigma \rightarrow \Sigma_{\rm eff}(V)$
  Quenched sickness: $\Sigma_{\rm eff} \propto \log(V)$

Matching QCD with chiral effective theory at NLO

$$\begin{aligned}
\mathrm{QCD:}\ \hat{\tau}_\nu(\hat{\lambda}_{\mathrm{min}},\hat{\lambda}_{\mathrm{max}}) &= 2\hat{m}\int_{\hat{\lambda}_{\mathrm{min}}}^{\hat{\lambda}_{\mathrm{max}}} \frac{1}{\hat{\lambda}^2+\hat{m}^2}\hat{\rho}_\nu(\hat{\lambda})d\hat{\lambda} \\
\mathrm{Eff.Th.:}\ \tilde{\tau}_\nu &= 2\Sigma_{\mathrm{eff}}\mu\int_{\zeta_{\mathrm{min}}}^{\zeta_{\mathrm{max}}} \frac{1}{\zeta^2+\mu^2}\overline{\rho}_\nu(\zeta)d\zeta
\end{aligned}$$

Our observable: $\hat{\tau}_{\nu_1}(\lambda_{\mathrm{min}1},\infty)-\hat{\tau}_{\nu_2}(\lambda_{\mathrm{min}2},\infty)$ $\rightarrow$ extract Σ_{eff}

- $\hat{\chi}_\nu = \hat{\tau}_\nu(0,\infty) = \hat{\tau}_\nu(\lambda_{\min},\infty) + 2\hat{\Sigma}_{\rm eff}\mu \int_0^{\zeta_{\min}} \frac{1}{\zeta^2+\mu^2}\overline{\rho}_\nu(\zeta) d\zeta$

$$\begin{aligned}
\mathrm{QCD}: \ \hat{\chi}_\nu &= 2\hat{m}\int_0^\infty \frac{1}{\hat{\lambda}^2+\hat{m}^2}\hat{\rho}_\nu(\hat{\lambda})d\hat{\lambda} \\
\mathrm{Eff.Th.}: \ \frac{\tilde{\chi}_\nu(\mu)}{\Sigma_{\rm eff}} &= \mu\left(I_\nu(\mu)K_\nu(\mu) + I_{\nu+1}(\mu)K_{\nu-1}(\mu)\right)
\end{aligned}$$

$(\mu = \hat{m}\Sigma_{eff} V)$

Osborn et al (1999), Damgaard et al (1999,2002)

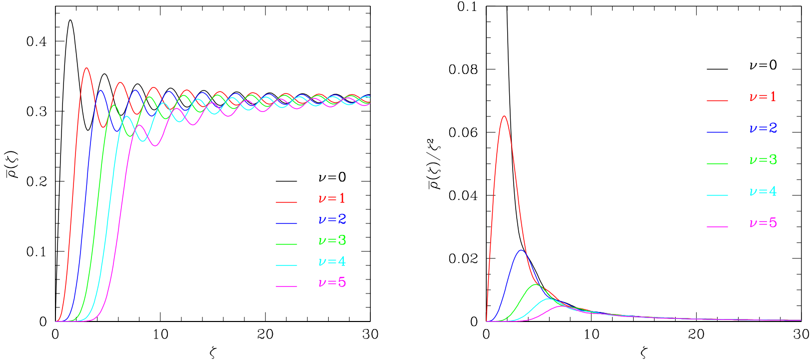
- Leutwyler-Smilga sum rule:

$$\text{QCD}: \quad \left(\frac{\hat{\chi}_{\nu_1} - \hat{\chi}_{\nu_2}}{\hat{\chi}_{\nu_3} - \hat{\chi}_{\nu_4}}\right)_{\mu=0}$$

$$\text{Eff.Th.}: \quad \left(\frac{\tilde{\chi}_{\nu_1} - \tilde{\chi}_{\nu_2}}{\tilde{\chi}_{\nu_3} - \tilde{\chi}_{\nu_4}}\right)_{\mu=0} = \frac{(\nu_1 - \nu_2)\nu_3\nu_4}{(\nu_3 - \nu_4)\nu_1\nu_2}$$

$\longrightarrow$ parameter-free predictions

What we expect from the effective theory:



$\rightarrow$ Large contributions to the integral from the region $\zeta \lesssim 20$

Numerical results

Simulation parameters:

latt.	β	L/a	L (fm)	$N_{\rm cfg}$	$N_{\rm cfg}^{\nu}$	am
c1	5.8458	12	1.49 fm	672	119, 205, 155, 104, 51, 29	0.001, 0.003, 0.008, 0.012 ,0.0016
c2	5.8458	16	1.98 fm	488	49, 69, 82, 72, 50, 54	0.000316, 0.000949, 0.00253, 0.00380, 0.00506
c3	6.0	16	1.49 fm	418	74, 137, 101, 62, 27, 12	0.000612, 0.00184, 0.00490, 0.00735, 0.00980

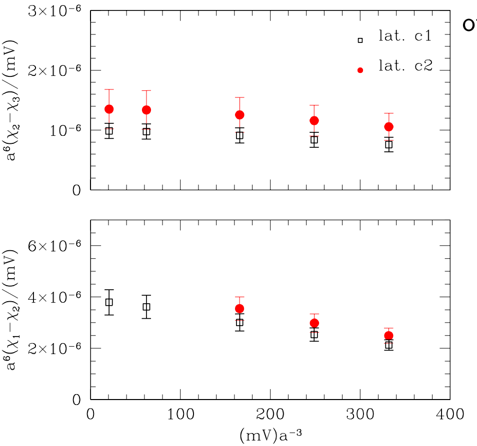
- masses and volumes chosen such that $\frac{mV}{\hat{Z}_S r_0^3} = \text{constant}$

  $\hat{Z}_S$: renormalisation factor of the RGI scalar density

  P. Hernández, K. Jansen, L.Lellouch and H. Wittig (2001), J. Wennekers and H. Wittig (2005)

- $0.07 \lesssim (\hat{m}\Sigma_{\rm eff} V) \lesssim 1.2$
- $n = 20$ low modes extracted $\rightarrow (\langle|\lambda_{20}|\rangle\Sigma_{\rm eff} V) > 20$

Finite-size scaling



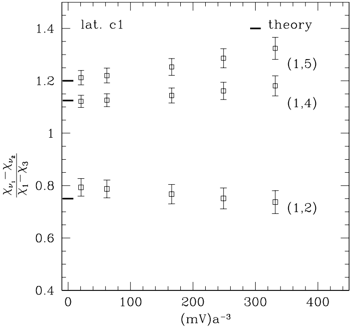
$a^3(\chi_{\nu_1} - \chi_{\nu_2})$ as a function of $(mV)a^{-3}$

- at leading order, we expect $a^3(\chi_{\nu_1} - \chi_{\nu_2})$ to be a function of (mV)
- within our precision, we are not sensitive to NLO corrections
- $V_{\mathrm{c2}}/V_{\mathrm{c1}} \simeq 3$ non-trivial verification of finite-size-scaling
- similar behaviour for higher topologies

Leutwyler-Smilga sum rules

(QCD) (effective theory)

$$\frac{\chi_{\nu_1} - \chi_{\nu_2}}{\chi_{\nu_3} - \chi_{\nu_4}} \quad \leftrightarrow \quad \left(\frac{\tilde{\chi}_{\nu_1}(\mu) - \tilde{\chi}_{\nu_2}(\mu)}{\tilde{\chi}_{\nu_3}(\mu) - \tilde{\chi}_{\nu_4}(\mu)} \right)_{\mu=0} = \frac{(\nu_1 - \nu_2)\nu_3\nu_4}{(\nu_3 - \nu_4)\nu_1\nu_2}$$

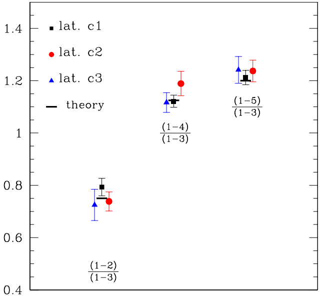


- very mild mass dependence

Leutwyler-Smilga sum rules

(QCD) (effective theory)

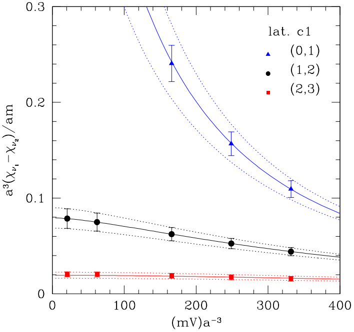
$$\frac{\chi_{\nu_1} - \chi_{\nu_2}}{\chi_{\nu_3} - \chi_{\nu_4}} \quad \leftrightarrow \quad \left(\frac{\tilde{\chi}_{\nu_1}(\mu) - \tilde{\chi}_{\nu_2}(\mu)}{\tilde{\chi}_{\nu_3}(\mu) - \tilde{\chi}_{\nu_4}(\mu)} \right)_{\mu=0} = \frac{(\nu_1 - \nu_2)\nu_3\nu_4}{(\nu_3 - \nu_4)\nu_1\nu_2}$$



- summary for the three lattices, at the lightest quark mass at our disposal
- the topology dependence in (quenched) QCD is well reproduced by chiral effective theory

Extraction of the chiral condensate

$(\tau_{\nu_1} - \tau_{\nu_2}) \rightarrow$ one-parameter fit to chiral perturbation theory $\rightarrow \Sigma_{\text{eff}}(V)$



run	$\nu_1 - \nu_2$	$a^3\Sigma_{\rm eff}/Z_S$
c1	0-1	0.0040(6)
	1-2	0.0039(3)
	2-3	0.0034(3)
c2	0-1	0.0035(8)
	1-2	0.0049(9)
	2-3	0.0040(5)
c3	0-1	0.0015(3)
	1-2	0.00178(18)
	2-3	0.00188(12)

RGI condensate: $\hat{\Sigma}_{\rm eff} = \hat{Z}_s \Sigma_{\rm eff}$

(c1) :$\hat{\Sigma}_{\rm eff}(L = 1.5 \text{ fm})\mathrm{r}_0^3 = 0.33(3)$
(c2) :$\hat{\Sigma}_{\rm eff}(L = 2.0 \text{ fm})\mathrm{r}_0^3 = 0.34(5)$
(c3) :$\hat{\Sigma}_{\rm eff}(L = 1.5 \text{ fm})\mathrm{r}_0^3 = 0.29(3)$

Finite volume effects and lattice artefacts below statistical uncertainty

Conversion to $\overline{\rm MS}$ scheme: $\overline{m}_{\overline{\rm MS}}(2 \text{ GeV})/\mathrm{M} = 0.72076$

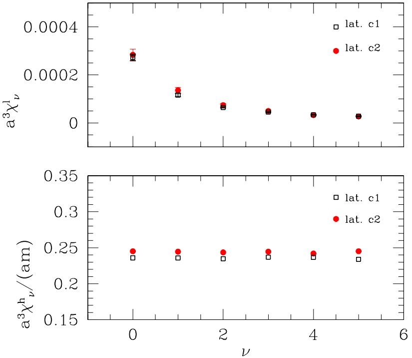
ALPHA collaboration (2000)

(c3): $\Sigma_{\overline{\rm MS}}(2\mathrm{GeV}) = (290 \pm 11\mathrm{MeV})^3$

Conclusions

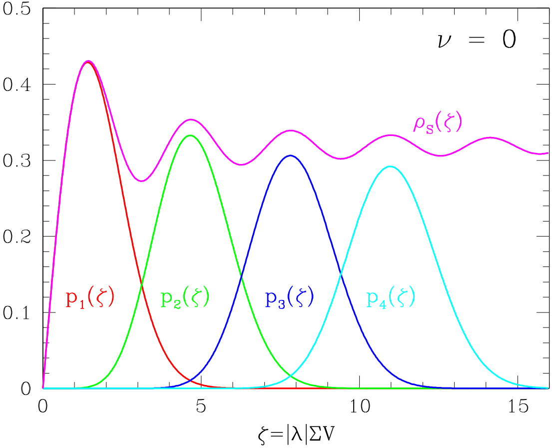
- The functional form of quenched QCD chiral condensate is in good agreement with chiral effective theory: volume, mass and topology dependence well reproduced (for $L = 1.5 - 2$ fm, $\hat{m}\Sigma V \leq 1$)
- close to the chiral limit good agreement with the first Leutwyler and Smilga sum rule
- finite-size scaling is verified $\longrightarrow$ spontaneous symmetry breaking can be investigated at finite volume $\longrightarrow$ extraction of $\Sigma_{\rm eff}$
- it is possible to introduce a stable numerical estimator for the condensate in the finite volume regime of QCD
  $\longrightarrow$ other choices for the observables can be tested
- lattice QCD yields a theoretically clean framework to test low-energy properties from first principles and extract low-energy constants (see P. Hernandez talk)

Topology and volume dependence of heavy and light contributions

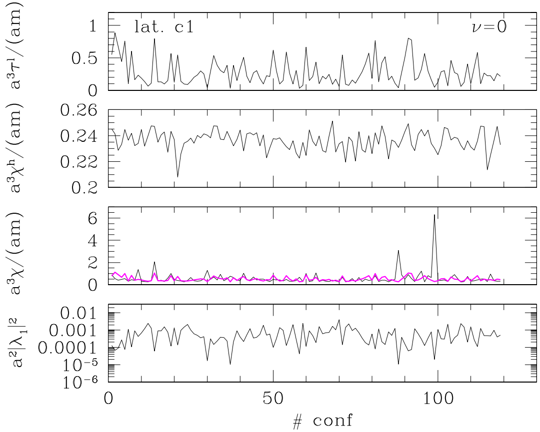


$$\rho_s(\zeta) = \zeta^{2(|\nu|+N_f)+1}$$

$\rightarrow$ arbitrarily small eigenvalues of D can occur, with probability decreasing with ν and N_f



Monte Carlo history for lattice c1, $am = 0.008$



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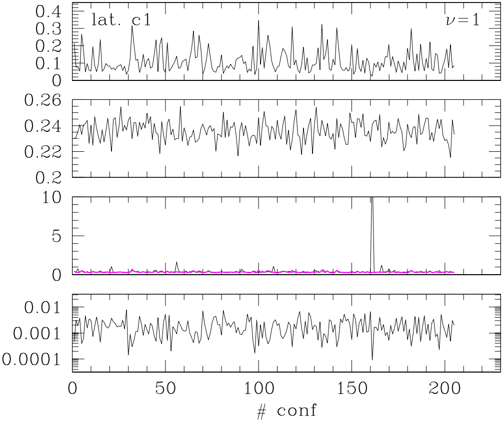
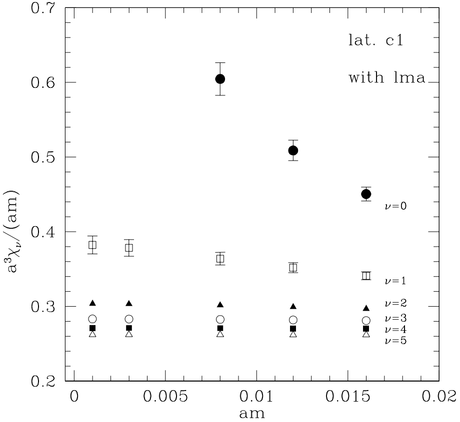


Variance reduction through low-mode averaging: lattice c1



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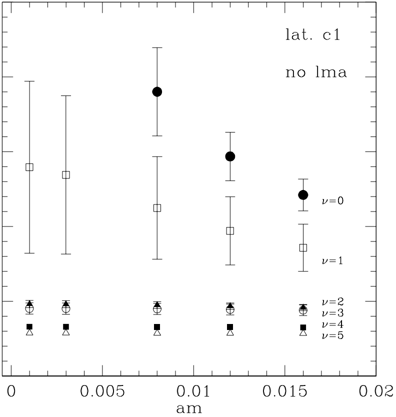

- The variance reduction is more effective at lower topologies
- For $\nu = 0$: good distribution only for the three heaviest masses

NLO prediction:

$$\mu\Sigma_\nu^{\mathrm{NLO}}(\mu) = \mu_{\mathrm{eff}}\Sigma_\nu^{\mathrm{LO}}(\mu_{\mathrm{eff}}); \qquad \mu_{\mathrm{eff}} = m\Sigma_{\mathrm{eff}}V$$

Damgaard et al (2002)

$$\left.\frac{2\nu\hat{\Sigma}_\nu^{\mathrm{NLO}}(m\Sigma V)}{\hat{m}V}\right|_{m=0} = \Sigma_{\mathrm{eff}}^2(V)$$

$$\frac{\Sigma_{\mathrm{eff}}(V_1)}{\Sigma_{\mathrm{eff}}(V_2)} = 1 + \frac{1}{3F^2}\left\{\frac{m_0^2}{2\pi^2}\ln\left(\frac{L_1}{L_2}\right) - \beta_1\alpha\left(\frac{1}{L_1^2} - \frac{1}{L_2^2}\right)\right\}$$

F pseudoscalar decay constant, m_0 singlet mass, α singlet kinetic term, β_1 shape coefficient